Broadcast in radio networks: time vs. energy tradeoffs. *

Marek Klonowski, Dominik Pająk

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Abstract

In wireless networks, consisting of battery-powered devices, energy is a costly resource and most of it is spent on transmitting and receiving messages. The goal of the paper is to study tradeoffs between time and energy complexity of broadcast problem in multi-hop radio networks. Broadcast is a problem where a message needs to be transmitted from one node to all other nodes of the network. We consider a model where the topology of the network is unknown and if two neighbors of a station are transmitting in the same discrete time slot, then the signals collide and the receiver cannot distinguish the collided signals from silence.

We study algorithms that can work under limited energy measured as the maximum number of transmissions by a single station. We observe that existing, time efficient, algorithms are not optimized with respect to energy expenditure.

On the positive side we present and analyze two randomized energy-efficient algorithms. Our first algorithm works in time $O((D+\varphi)\cdot n^{1/\varphi}\cdot\varphi)$ with high probability and uses $O(\varphi)$ energy per station for any $\varphi \leq \log n/(2\log\log n)$ and for any graph with n nodes and diameter D. Our second algorithm works in time $O((D+\log n)\log n)$ with high probability and uses $O(\log n/\log\log n)$ energy.

We prove that our algorithms are almost time-optimal for given energy limits by constructing lower bound on time of $\Omega(n^{1/\varphi}\cdot\varphi)$ for some graphs with constant diameter. The lower bound shows also that any algorithm working in polylogaritmic time in n for all graphs needs energy $\Omega(\log n/\log\log n)$.

1 Introduction

The problem of broadcast consists in delivering a single message from a source to all the nodes of a communication network. In multi-hop networks, neighboring stations can send messages to each other but when two neighbors of one station are sending at the same time, then these transmissions interfere and the messages are not delivered. Such a situation is in our model indistinguishable from silence (no collision detection). That is a station can locally distinguish only the case where exactly one neighbor transmits from all other situations. The broadcast problem is fundamental in radio networks because it can be used to learn the topology of the network or as a subprocedure to other, more complex problems, like multi-message broadcast or gossiping [8]. It also allows to emulate single-hop (networks where any two stations can exchange messages) algorithms in multi-hop networks [3].

Two most important parameters of an algorithm in radio networks is the time complexity, measured as the number of steps necessary to complete the execution, and energy complexity, which is the maximum number of rounds in which a station is transmitting. For single-hop networks both time and energy complexity of broadcast algorithms has been well studied. However, in the more general case, for multi-hop networks, only time complexity was analyzed. We aim at filling this gap and showing algorithms that minimize the time as well as the energy in multi-hop networks. We want to also show a tradeoff between time and energy for broadcasting protocols. This will allow greater flexibility when designing algorithms by decreasing the maximum energy expenditure of a station at a cost of the runtime of the algorithm. Clearly, minimizing the energy cost can sometimes be a critical aspect of real-life systems as they are often composed of small, cheap, battery-powered devices whose batteries cannot be easily

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recharged or replaced. For such systems it may be reasonable to sacrifice time and save some energy of the stations.

1.1 Model and Problem Statement

In this paper we consider a radio network represented as an undirected, connected graph $G = (V, E)$, where the nodes symbolize stations and the edges are bidirectional communication links between them. By n we denote the number of nodes of the graph and by D its diameter. We assume that the nodes are given value of n and the energy limit φ. The stations do not know the topology of the network, are identical and do not have any labels. Symmetry between the stations can be in our model broken as the stations have access to independent sources of random bits.

Time is divided into discrete *rounds* and all the stations know the number of the current round. The model is synchronous and in each round each node either transmits a message or listens. Each station receives a packet from its neighbors only if it listens in a given round and **exactly one** of its neighbors is transmitting. We say that a Single occurs in such a round. If zero or more than one neighbor of v transmits, then v receives no message.

The single-message broadcast problem is defined as follows. Initially some node, called *originator* has a message and the goal is to deliver this message to all the nodes in the network. We assume that the nodes that have not received the message are not allowed to make any transmissions.

Energy metrics We define e_v, an *energetic effort* of a station $v \in V$, as the number of rounds when v transmitted. Note that both successful as well as unsuccessful (due to collisions) transmissions count. We will say that algorithm uses energy at most E if $\max_{v \in V} e_v \leq E$. In other words we aim at limiting the energetic expenditure of **all** stations that are present in the network since we need all stations working. We will consider Monte Carlo algorithms using energy E and time T working with probability p which will be understood that the algorithm always terminates after at most T steps and each station uses at most E energy and the broadcast is successful with probability at least p. The same definition was used for example in [16, 21, 26].

In some of the existing protocols the energy expenditure is a random variable and in order to compare it with our solutions we will analyse their energy metric defined as the expected maximum amount of energy spent by a station: $E[\max_{v \in V} e_v]$. Such a definition was used for example in [20]. Note that in some models in single-hop networks, also listening to the channel costs energy (see e.g., [5, 7]).

Notation For any k, set of integers $\{1, 2, \ldots, k\}$ is denoted by $[k]$. By $\log x$ we denote logarithm at base 2 of x and by $\ln x$ the natural logarithm.

1.2 Our results

In this paper we present two energy-efficient broadcast algorithms and a lower bound. We first show an universal algorithm BB-Broadcast in which the available energy is a parameter. This allows us to obtain a complete tradeoff between time and energy for any fixed energy between constant and $\log n / \log \log n$. The second algorithm GD-Broadcast is a modification of the broadcast algorithm from [3]. The goal of the second algorithm is to perform broadcast in the same (almost-optimal) time as in [3] but reduce its energy complexity. The obtained algorithm has energy complexity $O(\log n / \log \log n)$ which we will later show to be the minimum energy complexity for any algorithm with time polylogarithmic in n. We also present a lower bound which shows that our algorithms are almost time-optimal for graphs with constant diameter.

Time	Energy	Result	Remarks
$O((D+\log n)\log n)$	$O\left(\frac{\log n}{\log\log n}\right)$	Thm 7	
$O\left(\left(D+\frac{\log n}{\alpha\log\log n}\right)\frac{\log^{1+\alpha} n}{(\alpha-1)\log\log n}\right)$	$O\left(\frac{\log n}{\alpha\log\log n}\right)$	Thm 4	$\alpha > 1$
$\Omega\left(\frac{\log^{1+\alpha} n}{\alpha\log\log n}\right)$	$\frac{\log n}{\alpha\log\log n}$	Corr 9	already for constant D
$O((D+\varphi)\cdot n^{1/\varphi}\cdot\varphi)$	$O(\varphi)$	Thm 4	$\varphi \in o(\log n/\log\log n)$
$\Omega(n^{1/\varphi}\cdot\varphi)$	φ	Corr 9	already for constant D

Table 1: Summary of our results. For example by setting $\varphi = \frac{\sqrt{\log n}}{\log\log n}$ shows that there exists an algorithm with energy $O(\sqrt{\log n})$ and time $\widetilde{O}(D\cdot 2^{\sqrt{\log n}})$ whereas time $\Omega(2^{\sqrt{\log n}})$ is required for such energy for some graphs with constant diameter. Similarly for energy $O(\log\log n)$, our algorithm uses $\widetilde{O}(D\cdot n^{1/\log\log n})$ time steps.

LOCAL BROADCAST

Time	Energy
$O(\log^2 n)$	$O\left(\frac{\log n}{\log\log n}\right)$
$\Omega(\log^2 n)$	dowolna
$O\left(\frac{\log^{2+\alpha} n}{(\alpha\log\log n)^2}\right)$	$O\left(\frac{\log n}{\alpha\log\log n}\right)$
$\Omega\left(\frac{\log^{1+\alpha} n}{\alpha\log\log n}\right)$	$\frac{\log n}{\alpha\log\log n}$
$O(n^{1/\varphi}\cdot\varphi^2)$	$O(\varphi)$
$\Omega(n^{1/\varphi}\cdot\varphi)$	φ

1.3 Related work

Broadcast problem The first randomized broadcast protocol presented by Bar-Yehuda *et al.* [4] for the model without collision detection works in any multi-hop network in time $O(D\log n + \log^2 n)$ with high probability. It is based on Decay procedure from [3] extensively used also in many other papers. Improved protocols with expected time $O(D\log(n/D) + \log^2 n)$ have been independently proposed by Czumaj and Rytter [9] and Kowalski and Pelc [23]. Those results are optimal due to the lower bounds $\Omega(\log^2 n)$ shown by Alon *et al.* [1] and $\Omega(D\log(n/D))$ by Kushilevitz and Mansour [25]. In the model with known topology Gasieniec *et al.* [13] showed a randomized algorithm with time $O(D+\log^2 n)$ and Kowalski and Pelc [24] showed a deterministic algorithm with the same complexity. The algorithms are time-optimal since the bound $\Omega(\log^2 n)$ by Alon *et al.* [1] holds also for the known topology. In some papers, broadcast in specific graph classes, like line [10] or planar graphs [11, 13] was considered.

In the model with collision detection (Receiver-CD), where listening stations can distinguish between silence and collision, Ghaffari *et al.* [14] presented randomized protocol with expected time $O(D+\log^6 n)$ for networks with unknown topology.

Energy efficient protocols Many problems have been discussed in the context of energy complexity in the single hop networks. Bender *et al.* [5] presented algorithm for contention resolution (where each station has to transmit a message to the shared channel) achieving constant throughput using on average only $O(1)$ energy for transmitting and $O(\log(\log^* N))$ for listening per station, where N is the (known) size of the devices' ID space. Problems of leader election, size approximation and census problem have been studied by Chang *et al.* [16], who studied models of Receiver-CD and Sender-CD (where a station that sends the message receives feedback whether the transmission was successful but receiver cannot detect collisions). For leader election in randomized settings with unknown number of stations n, they proved surprising gap between energy $\Theta(\log^* n)$ in Sender-CD and $\Theta(\log(\log^* n))$ in Receiver-CD. In deterministic setting, the exponential gap turned out to be in the reverse direction: $\Theta(\log N)$ for Receiver-CD and $\Theta(\log\log N)$ for Sender-CD.

For [17] Jurdzinski *et al.* proved lower bound $\Omega(\log\log n/\log\log\log n)$ for energy complexity of leader election in Sender-CD. The same authors presented algorithm with energy complexity $O(\sqrt{\log n})$ [19]. In [18] Jurdzinski *et al.* presented size approximation with polylogarithmic execution time and energy complexity $o(\log\log n)$. In [20] Kardas *et al.* constructed an algorithm in Strong-CD model (both sender and receiver can detect collisions) with $O(\log\log\log n)$ expected energy and $O(\log^{\epsilon} n)$ time for $\epsilon > 0$.

In Strong-CD the stations can assign themselves unique identifiers in range $[n]$, using $O(\log\log n)$ energy [6, 27].

2 Analysis of classic protocols

In this section we discuss energy complexity of classic broadcasting protocols with optimal (or close to optimal) execution time. We will prove that they were not optimized with respect to energy complexity. All proofs from this section are deferred to Appendix.

In [4] Bar-Yehuda, Goldreich and Itai presented an algorithm working in time $O(D\log n + \log^2 n)$. This algorithm uses procedure Decay which is also utilized in many other protocols.

Fact 1. *The energy complexity of Bar-Yehuda, Goldreich and Itai protocol is* $\Theta(\log n\log\log n)$.

Czumaj and Rytter [9] presented an asymptotically time-optimal algorithm working under the assumption that $D = \Omega(\log^3 n)$ in time $O(D\log(n/D) + \log^2 n)$.

Fact 2. *The energy complexity of Czumaj and Rytter protocol is* $\Omega(D)$ *for* $D = \Omega(\log^3 n)$.

The same (optimal) time complexity achieve algorithm by Kowalski and Pelc [23].

Fact 3. *The energy complexity of Kowalski and Pelc protocol is* $\Omega(D)$ *if* $D = \Omega(n^{2/3})$ *and* $\Omega(\log n\log\log n)$ *otherwise.*

Facts 1, 2 and 3 are motivation for our study. We observed that the existing algorithms cannot work using energy lower than $\Omega(\log n\log\log n)$. Moreover it is not clear if it is possible to directly convert these algorithms in order to use them under any energy limit.

All of the existing procedures use the following framework. Each station in i-th round after receiving the message transmits with some probability p_i. Algorithm, such that all station transmit with the same probability in a given round are called *uniform*. One approach to design energy-efficient algorithms based on the existing ones could be to decrease the probabilities p_i. Observe however that if $p_i < 1/\hat{n}$ (where $\hat{n}$ is the number of stations trying to transmit to the same node) then $\Pr[\mathsf{Single}] \approx p_i\hat{n}$ and decreasing the probabilities p_i by a factor of 2 decreases the probability of Single by a factor of close to two which in turn increases the runtime of the algorithm. If the runtime is larger by a factor close to 2 and the probabilities are smaller by the same factor then the ratio between the total energy spent after and before the modification is close to 1. Hence the gain in terms of energy efficiency from such modification is probably small. To develop energy efficient algorithms for multi-hop broadcast we will need a different approach – we will propose an approach based on a classical problem Balls into Bins.

3 Energy-efficient broadcast algorithms

In this section we present two new algorithms. The first, BB-Broadcast ("Balls-into-Bins Broadcast") can work with arbitrarily small (also constant) available energy. Of course smaller energy leads to higher running time. The second algorithm GD-Broadcast ("Green Decay Broadcast") is built based on classical Broadcast by Bar-Yehuda et al. [4]. Our GD-Broadcast algorithm works in the same asymptotic time as the original one, but has reduced energy complexity $O(\log n/\log\log n)$. We will later show that any algorithm that works in polylogarithmic time on graphs with constant diameter needs at least this much energy.

3.1 Balls into Bins

Here we introduce a subprocedure Balls-into-Bins(k) that will be used in both our algorithms. In this subprocedure each participating station transmitts in one, randomly chosen out of k time slots. Balls into Bins (called also Random Mapping) is a classical concept in probability theory with applications in load balancing where the studied value is usually the maximum number of balls in any bin (e.g. [22]). In our setting, balls correspond to transmissions by the stations and bins are the time slots, hence we will be interested whether some bin will contain exactly one ball as this will correspond to a successful transmission.

```
1 Let t be the first time slot of the subprocedure
2 Choose a random number i ∈ [0, 1, .., k − 1]
3 Transmit in slot t + i
```

Algorithm 1: Balls-into-Bins(k)

In the following lemma we bound the probability that in a procedure Balls-into-Bins(k) there is a slot that is chosen by exactly one station.

Lemma 1. *If m neighbors of any fixed node v are performing procedure* Balls-into-Bins(k) *with $k = 24\lceil n^{1/\varphi}\rceil + 1$ if $1 \leq \varphi < \frac{\log n}{\log\log n}$ and if $1 \leq m \leq 12/\varphi \cdot n^{1/\varphi}\ln n$ then v receives the message with probability at least $1 - \frac{1}{2n^{1/\varphi}}$ for sufficiently large n.*

Proof. The probability that v receives the message is equal to the probability that when throwing m balls into k bins (independently and uniformly at random) at least one bin contains a single ball.

The case $m = 1$ is trivial. Consider case $1 < m < 12\lceil n^{1/(2\varphi)}\rceil$. Take the two last balls and observe that the probability that (at the moment when each of the balls is thrown) either of them lands in one of the bins that have not been already occupied is at least

$$1 - \left(\frac{12\lceil n^{1/(2\varphi)}\rceil}{k}\right)^2.$$

With probability $1/k$ the last two balls collide hence with probability at least

$$1 - \left(\frac{12\left\lceil n^{1/(2\varphi)}\right\rceil}{k}\right)^2 - \frac{1}{k} \geq 1 - \frac{1}{2n^{1/\varphi}},$$

at least one of the two last balls ends up as the only ball in one of the bins. For $12/\varphi \cdot n^{1/\varphi}\ln n \geq m \geq 12\lceil n^{1/(2\varphi)}\rceil$ we use [21, Lemma 4] and obtain

$$\begin{aligned}
\mathbf{Pr}\left[X = 0\right] &\leq \exp\left(-\frac{m}{2}\left(1 - \frac{1}{k}\right)^{2m-2}\right) = \exp\left(-\frac{m}{2}\left(\left(1 - \frac{1}{k}\right)^{k-1}\right)^{\frac{2m-2}{k-1}}\right) \\
&\leq \exp\left(-\frac{m}{2}\left(\exp\left(-\frac{2m-2}{k-1}\right)\right)\right) = f(m,k)
\end{aligned}$$

The derivative of $f(n,k)$ with respect to m is equal to:

$$\exp\left(-\frac{m}{2}\left(\exp\left(-\frac{2m-2}{k-1}\right)\right) - \left(\exp\left(-\frac{2m-2}{k-1}\right)\right)\right) \cdot \left(\frac{m}{k-1} - \frac{1}{2}\right),$$

The function has a single minimum for $2m = k-1$ and because the derivative is negative for $2m < k-1$, the maximum value is attained in one of the endpoints of the considered interval. Knowing that $\varphi < \frac{\log n}{\log\log n}$, we get

$$f(12\lceil n^{1/(2\varphi)}\rceil, 24\lceil n^{1/\varphi}\rceil + 1) \leq \exp\left(-6n^{1/\varphi}\right) \leq \frac{1}{2n^{1/\varphi}}.$$

$$f(12\lceil 1/\varphi \cdot n^{1/\varphi}\ln n\rceil, 24\lceil n^{1/\varphi}\rceil + 1) \leq \exp\left(-6/\varphi \cdot n^{1/\varphi}\ln n \exp\left(-\frac{24\lceil 1/\varphi \cdot n^{1/\varphi}\ln n\rceil}{24\lceil n^{1/\varphi}\rceil}\right)\right) \leq \frac{1}{2n^{1/\varphi}},$$

□

This lemma cannot be easily improved by more than a constant factor, since in Balls into Bins problem if we want each bin to contain at least b balls, the total expected needed number of needed balls is close to $k \log k + (b-1) \cdot k \log \log k$ [28]. And the concentration bound is very strong – having $k \log k + k \log \log k + ak$ balls, the probability that each bin contains at least two balls is for large a close to $e^{-e^{-a}}$ [12].

3.2 Balls into Bins Broadcast

Goal of this section is to design an algorithm that uses at most $O(\varphi)$ transmissions and has time complexity roughly $O(D \cdot n^{1/\varphi} \cdot \varphi)$ for any $\varphi > 0$. The idea of the algorithm is as follows. We consider the algorithm from the perspective of a fixed node u at the first step when at least one of its neighbors v has the message. The goal is to deliver the message to u with high probability in time $O(n^{1/\varphi})$. Each participating station (neighbor of u that has the message) choses a phase being a number chosen according to geometric distribution with parameter $\varphi/n^{1/\varphi}$. We will show that regardless of how many participants are there, some value will be chosen by at most $O(n^{1/\varphi}/\varphi \cdot \ln n)$ stations. Each phase is a Balls-into-Bins$(n^{1/\varphi})$ procedure. We already know that if the number of participants is $O(n^{1/\varphi}/\varphi \cdot \ln n)$, then the failure probability of such procedure is at most $n^{-1/\varphi}$. Then, repeating it φ times will give us high probability. This intuition is further specified in the following pseudocode and formalized in the next two lemmas.

```
1  a ← ⌈ φ log n / (log n − φ log φ) ⌉
2  k ← 24⌈n^{1/φ}⌉ + 1
3  t_ph ← ak
4  Wait until receiving the message;
5  repeat
6  |  Wait until (Time mod t_ph) = 0
7  |  Choose a number x ~ Geo(φ/n^{1/φ})
8  |  Skip (min{x, a} − 1) · k rounds
9  |  Balls-into-Bins(k)
10 until 2φ times;
```

Algorithm 2: BB-Broadcast(φ)

The following lemma shows that, regardless of how many stations execute line 7 of the Algorithm 2 in parallel, some number is chosen by at most $12/\varphi \cdot n^{1/\varphi} \ln n$ stations.

Lemma 2. *For any $1 \leq \varphi < \frac{\log n}{\log \log n}$ and for $a = \left\lceil \frac{\varphi \log n}{\log n - \varphi \log \varphi} \right\rceil$, if $1 \leq \hat{n} \leq n$ random variables $X_1, X_2, \ldots, X_{\hat{n}}$ are chosen from $\mathsf{Geo}(\varphi/(\mathsf{n}^{1/\varphi}))$ then with probability at least $1 - 2/n^2$ among $Y_i = \min\{X_i, a\}$ there is a number $y \in \{1, 2, \ldots, a\}$ chosen at least once and at most $12/\varphi \cdot n^{1/\varphi} \ln n$ times by Y variables.*

Proof. If $\hat{n} \leq 12/\varphi n^{1/\varphi} \ln n$ then the statement follows. Consider the opposite case. We have a set of $\hat{n}$ independent identically distributed random variables $X_1, \ldots, X_{\hat{n}} \sim \mathsf{Geo}(\varphi/\mathsf{n}^{1/\varphi})$. Observe that since $\varphi < \frac{\log n}{\log \log n}$ then $\varphi \log \varphi < \log n$ and if we denote $b = \left(\frac{1}{\varphi} - \frac{\log \varphi}{\log n}\right)$ then $b > 0$. Observe that $a = \lceil 1/b \rceil$ (line 1 of pseudocode). We have for each $j \in \{1, 2, \ldots, \hat{n}\}$:

$$\mathbf{Pr}\left[X_j > i\right] = \varphi^i n^{-i/\varphi} = n^{-i\left(\frac{1}{\varphi} - \frac{\log \varphi}{\log n}\right)} = n^{-i \cdot b}, \quad \text{for any } i = 1, 2, \ldots, a.$$

Thus $\mathbf{E}\left[|j : X_j > i|\right] = \hat{n} \cdot n^{-i \cdot b}$. We know that $\hat{n} > 12/\varphi \cdot n^{1/\varphi} \ln n$ and $n^b = n^{1/\varphi}/\varphi$ thus $\hat{n} > 4n^b \ln n$. Take the smallest i^* such that $\hat{n}/n^{i^* \cdot b} < 4n^b \ln n$. Since $b > 0$, such i^* exists and $i^* \geq 1$ and moreover since $\hat{n} \leq n$ then $i^* \leq 1/b \leq a$. Using the minimality of i^* we can write $\hat{n} = n^{i^* \cdot b} \cdot r$, where $4n^b \ln n \geq r \geq 4 \ln n$ We define the following variables

$$Z_j = \begin{cases} 1 & \text{if } X_j > i^*, \\ 0 & \text{otherwise.} \end{cases}$$

If $Z = \sum_{i=1}^{\hat{n}} Z_i$ then $\mathbf{E}[Z] = r \geq 4 \ln n$ and by Chernoff bound:

$$\mathbf{Pr}[Z = 0] \leq \mathbf{Pr}[Z \leq (1-1)\mathbf{E}[Z]] \leq e^{-\frac{\mathbf{E}[Z]}{2}} \leq \frac{1}{n^2},$$

$$\mathbf{Pr}\left[Z > 12/\varphi \cdot n^{1/\varphi} \ln n\right] = \mathbf{Pr}\left[Z > 12n^b \ln n\right] \leq \mathbf{Pr}[Z > 3\mathbf{E}[Z]] \leq e^{-\frac{2\mathbf{E}[Z]}{3}} \leq \frac{1}{n^2}.$$

Thus with probability at least $2/n^{-2}$ between 1 and $12/\varphi \cdot n^{1/\varphi} \ln n$ values of X variables are at least i^*. Hence there is some value of Y variables chosen at least once and at most $12/\varphi \cdot n^{1/\varphi} \ln n$ times. ☐

Take a node v (different from the originator) and consider the smallest step t in which some neighbor of v receives the message. We will consider the next 2φ complete phases that start after t. We say that a phase is successful (for v) if it results in a Single and delivers the message to v.

Lemma 3. *For any $1 \leq \varphi < \frac{\log n}{\log\log n}$ in* BB-Broadcast(φ)*:*

1. *each phase is successful with probability at least $1 - \frac{1}{2n^{1/\varphi}} - \frac{2}{n^2}$*
2. *with probability at least $1 - \frac{1}{n}$ for all the nodes some of the 2φ phases is successful.*

Proof. In any phase, by Lemma 2 with probability at least $1 - 2/n^{-2}$, some procedure Balls-into-Bins is executed by at most $12/\varphi \cdot n^{1/\varphi} \log n$ stations. In such a case by Lemma 1, procedure Balls-into-Bins obtains a Single with probability at least $\frac{1}{2n^{1/\varphi}}$. Hence, with probability at least $1 - \frac{1}{2n^{-1/\varphi}} - \frac{2}{n^2}$ the phase is successful (results in a Single in one of its time slots). Using independence, the probability that 2φ phases are unsuccessful is at most

$$\left(\frac{1}{2n^{1/\varphi}} + \frac{2}{n^2}\right)^{2\varphi} = \frac{1}{n^2}\left(\frac{1}{2} + \frac{2}{n^{2-1/\varphi}}\right)^{2\varphi} \leq \frac{1}{n^2},$$

Taking union bound over all n stations, we get that with probability at least $1 - n^{-1}$, some phase is successful at every node. ☐

Theorem 4. *For any $1 \leq \varphi < \frac{\log n}{\log\log n}$, if $n > 2^{64}$, then Algorithm* BB-Broadcast *completes broadcast*

1. *in time $O\left((D+\varphi) \cdot n^{1/\varphi} \cdot \frac{\varphi \log n}{\log n - \varphi \log \varphi}\right)$,*
2. *using at most 2φ energy per station,*
3. *with probability at least $1 - 2/n$.*

Proof. Claim 2 follows directly from the construction of the algorithm. By Lemma 3 we know that with probability at least $1 - 1/n$ each station eventually receives the message. We could bound the total number of phases until each station receives the message by $O(D \cdot \varphi)$. We want however a slightly better bound of $O(D+\varphi)$ using the fact that on average only a constant number of phases is sufficient to deliver the message from one node to its neighbor.

Take any vertex v and consider the shortest path $\mathcal{P} = (u, v_1, v_2, \ldots, v_{D_v - 1}, v)$ of length $D_v \leq D$ from the originator u to v. Let us introduce random variables $X_i^{(v)}$ as the number of phases between the reception of the message by i-th node on path $\mathcal{P}$ and the reception of the message by $i+1$-st node on the path. We know by Lemma 3 that each phase is successful independently with probability at least $1 - \frac{1}{2n^{1/\varphi}} - \frac{2}{n^2} \geq 1 - \frac{1}{n^{1/\varphi}}$ and all $X_i^{(v)} \leq 2\varphi$ for all nodes v with probability at least $1 - 1/n$. We condition on the fact that $X_i^{(v)} \leq 2\varphi$ for all v and i. Observe that conditioned on this event, variable $X_v = \sum_{i=0}^{D_v} X_i^{(v)}$ is stochastically dominated by a negative binomial variable where we need D_v successes and probability of success is $1 - 1/n^{1/\varphi}$.

Assume first that $D_v > \log n$ and denote $X = \sum_{i=0}^{D_v} X_i^{(v)}$. We can use Chernoff bound for the sums of geometric variables (see e.g., [2, Theorem 1.14]):

$$\mathbf{Pr}\left[\sum_{i=0}^{D_v} X_i^{(v)} \geq 10\mathbf{E}[X_v]\right] \leq e^{4D_v} \leq \frac{1}{n^4}.$$

Observe that $\mathbf{E}[X] = D_v/(1-1/n^{1/\varphi})$ hence $10\mathbf{E}[X] = O(D_v)$. In the opposite case the concentration bound does not give us high probability so we have to calculate it in another way. Assume that $D_v \leq \log n$ and let $t = D_v + 6 \cdot \varphi$. If $n > 2^{64}$ we have that $\varphi \leq \log n/6$ hence $t \leq 2\log n$. We then have for any $0 \leq i < D'\varphi$:

$$\mathbf{Pr}\left[\sum_{i=1}^{D_v} X_i^{(v)} = t+i\right] \leq \binom{t+i}{6\varphi+i}\left(\frac{1}{n^{1/\varphi}}\right)^{6\varphi+i} \cdot \left(1-\frac{1}{n^{1/\varphi}}\right)^{D_v} \leq \frac{1}{n^6}\left(\frac{e\cdot(t+i)}{6\varphi+i}\right)^{6\varphi+i}\frac{1}{n^{i/\varphi}} \leq$$
$$\leq \frac{1}{n^6}\left(\frac{e\cdot t}{6\varphi}\right)^{6\varphi},$$

where the last inequality can be shown by a simple induction on i. We then have:

$$\mathbf{Pr}\left[\sum_{i=1}^{D_v} X_i^{(v)} = t+i\right] \leq \frac{1}{n^6}\left(\frac{e\cdot t}{6\varphi}\right)^{6\varphi} \leq \frac{1}{n^5},$$

because inequality $\left(\frac{e\cdot t}{6\varphi}\right)^{6\varphi} \leq \left(\frac{2e\log n}{6\varphi}\right)^{6\varphi} \leq n$ can be shown by substituting $\varphi = \alpha\frac{\log n}{\log\log n}$, for some $\alpha < 1$ and taking logarithm on both sides. We get

$$\sum_{i=0}^{(D_v-1)\varphi-6\varphi} \mathbf{Pr}\left[\sum_{i=1}^{D_v} X_i^{(v)} = t+i\right] \leq \frac{1}{n^4},$$

By taking a union bound in both case over all vertices v we get that with probability at most $1-1/n^3$, the number of phases each node receives the message is at most $O(D+\varphi)$, conditioned on the fact that all the nodes receive the message. Since we know that the latter event takes place with probability at least $1-1/n$ then with probability at least $1-2/n$ all the nodes receive the message and the total number of phases is $O(D+\varphi)$. Observe that the stations terminate the algorithm after additional time at most $\varphi \cdot t_{ph}$. Thus, since $t_{ph} \in O\left(n^{1/\varphi} \cdot \frac{\varphi\log n}{\log n - \varphi\log\varphi}\right)$, we obtain the desired result. □

Observe that if $\varphi \leq \frac{\log n}{2\log\log n}$ then $\frac{\varphi\log n}{\log n-\varphi\log\varphi} = \Theta(\varphi)$ and the complexity of the algorithm becomes $O((D+\varphi)n^{1/\varphi}\cdot\varphi)$.

3.3 Green-Decay

Algorithm BB-Broadcast can operate under a wide range of energy limits however for energy close to the upper bound $\log n/(\log\log n+1)$ guarantees only $O\left(\left(D+\frac{\log n}{\log\log n}\right)\cdot\frac{\log^2 n}{\log\log\log n}\right)$, which is slower than the algorithms from the literature. In this section we want to develop an algorithm that is less universal but achieves an almost-optimal time $O((D+\log n)\log n)$ using energy $O(\log n/\log\log n)$.

We will first present Green-Decay which is a simple modification of classical procedure Decay introduced in [3]. It will serve as a subprocedure to our energy-efficient algorithm GD-Broadcast. In original Decay, each station transmits for a number of rounds begin a geometric random variable. We note that instead of broadcasting in each round of the procedure it is sufficient to broadcast only in the last one. With this we save energy whilst the probability of success remains the same.

Transmit;
repeat
  $x \leftarrow 0$ or 1 with equal probability;
  if $x = 1$ **then**
    Transmit;
until $x = 1$ *but at most* k *times*;

Algorithm 3: Green-Decay(k)

The following theorem is analogous to [3, Theorem 1]. By inspection of the proof from [3] we observe that exactly the same proof works for Green-Decay. Furthermore it is easy to see that the modified procedure uses only constant energy.

Theorem 5. *If n neighbors of station v execute procedure* Green-Decay(k) *then the probability* $\mathbf{Pr}\,[\,k,n\,]$ *that v receives the message satisfies:*

1. $\mathbf{Pr}\,[\,\infty,n\,] = \frac{2}{3}$,
2. $\mathbf{Pr}\,[\,k,n\,] > \frac{1}{2}$, *for* $k \geq 2\log n$.

The high-level idea of GD-Broadcast is as follows. In standard Broadcast from [4] each station participates in $\Theta(\log n)$ Decay procedures and each Decay gives a constant probability of success. By simply replacing it with Green-Decay (that takes always at most a constant energy per participant) we can reduce the energy complexity from $\log n \log\log n$ to $\log n$. In order to reduce the energy complexity further we observe that a station does not necessarily need to participate in all the $\log n$ procedures Decay. If some number x of neighbors of v have the message and want to transmit it to v it is sufficient that in at least a constant fraction of the $\Theta(\log n)$ procedures Decay at least one among the x stations participate. In our algorithm each station participates in $\Theta(\log n/\log\log n)$ procedures Decay chosen at random. If x is sufficiently large, at least a constant fraction of procedures Decay will have at least one participant and the algorithm will work correctly. On the other hand if x is small we can use procedure Balls-into-Bins which gives a probability of success of order $1 - 1/\log n$, for $\varphi = \log n/\log\log n$. Hence for small x, $O(\log n/\log\log n)$ procedures Balls-into-Bins is sufficient to obtain the high probability of successful transmission. Our algorithm combines Green-Decay and Balls-into-Bins to cover both cases of small and large number of neighbors trying to deliver the message. One more difficulty we need to overcome is that the number of participating (i.e., holding the message) neighbors of v might increase over time.

```
ll ← ⌈log log n⌉
k ← 24⌈log n⌉ + 1
state ← new
Wait until receiving the message;
repeat
    Wait until (Time mod 3·k) = 0
    phase ← Time/(3·k) mod ll
    if phase = 0 and state = new then
        state ← normal
    endif
    if state = new then
        Balls-into-Bins(k)
        Skip k rounds
    else if phase = 0 then
        Skip k rounds
        Balls-into-Bins(k)
    endif
    if phase = 0 or state = new then
        if state = new then  state ← normal
        myPhase ← Random([0, 1, ..., ll − 1])
    endif
    if phase = myPhase then
        Green-Decay(k)
    endif
until k times;
```

Algorithm 4: GD-Broadcast

Observe that the algorithm executed by a node that received the message consists of $3k = 72\lceil\log n\rceil$ *phases*. Each phase consists of $t_{ph} = O(\log n)$ rounds. In the analysis we group $ll = \lceil\log\log n\rceil$ consecutive phases into an *epoch*.

Let $T(v)$ denote the time when v receives the message. Bounding the difference between $T(v)$ and $T(w)$ for any neighbors v and w is a key component in the analysis of the algorithm.

Lemma 6. *If $n > 16$ then for any two neighbors $v, w \in V$ and for any $1 \le x \le 2\lceil \log n \rceil$*

$$\mathbf{Pr}\left[\, |T(v) - T(w)| \ge (x+2) \cdot t_{ph} \,\right] \le 2^{-x}.$$

Proof. Without loss of generality assume that $T(v) < T(w)$. Consider $(x+2) \cdot t_{ph}$ steps, starting from $T(v)$, during which v executes algorithm GD-Broadcast. It performs at least $x+1$ complete phases. From the definition of the algorithm we can observe that station v executes $a < ll$ phases in state `new`, then it participates in b full epochs and finally $c < ll$ phases of the last (incomplete) epoch. Hence v participates altogether in the considered time interval in $b+2$ epochs. We will analyse the first epoch, consisting of a phases separately.

Observe that a station can finish its algorithm only at the end of a phase (see lines 6 and 25). Define by κ_i the number of neighbors of v participating in i-th epoch i.e. the number of stations that participate in the i-th epoch in all its phases in state `normal`.

1. Epochs $\{2, 3, \ldots, b+2\}$.
   (a) If $\kappa_i \ge 12\lceil \log n \rceil \log\log n$ then the probability that one fixed of the ll procedures Green-Decay is not executed by any station is at most
   $$\left(1 - \frac{1}{\lceil \log\log n \rceil}\right)^{12\lceil \log n \rceil \log\log n} \le n^{-12} \left(1 + \frac{1}{\lceil \log\log n \rceil - 1}\right)^{12\lceil \log n \rceil} \le n^{-3}, \qquad (1)$$
   where the inequality $\left(1 + \frac{1}{\lceil \log\log n \rceil - 1}\right)^{12\lceil \log n \rceil} \le n^{9}$ is true for $n > 16$. Hence with probability at least n^{-2} (by the Union Bound) each Green-Decay has at least one participant. But then by Theorem 5 each Green-Decay it is successful with probability at least $1/2$.
   (b) Fix any i and assume that $\kappa_i < 12\lceil \log n \rceil \log\log n$. Observe that exactly κ_i stations are taking part in Balls-into-Bins procedure (in line 16) which is executed by all stations in `normal` state always in the first phase in each epoch. Now using Lemma 1 if $\kappa_i < 12\lceil \log n \rceil \log\log n$, the failure probability of Balls-into-Bins is at most $1/(2\log n) \le 2^{-ll}$.
2. Epoch 1. Denote by κ the number of stations that execute procedure Balls-into-Bins in state `new` together with v (line 13). Observe that if $\kappa < 12\lceil \log n \rceil \log\log n$ then the argument is the same as in the previous case which gives success with probability at least $1 - 2^{-ll}$. In the opposite case all these κ stations choose a phase (line 21) and by (1) each of the a procedures Green-Decay is executed by at least one station with probability at least n^{-2} and hence successful with probability at least $1/2$.

We showed that if an epoch i has $\kappa_i \ge 12\lceil \log n \rceil \log\log n$ then the probability of success in each phase is at least $1/2 - n^{-2}$. And in the opposite case the probability of success of the entire epoch is at least $1 - 2^{-ll}$. Let y be the number of epochs i with $\kappa_i \ge 12\lceil \log n \rceil \log\log n$ and $z = x + 1 - y$. Then, using the independence of the phases, the failure probability is at most:

$$(1/2 + n^{-2})^{y \cdot ll} \cdot 2^{-ll \cdot z} = 2^{-x-1} \cdot (1 + 2 \cdot n^{-10})^{y} \le 2^{-x},$$

because $(1 + 2 \cdot n^{-2})^{y \cdot ll} \le 2$. Now we note that in both cases the probability of success in each considered epoch is at least 2^{-i} where i is the number of phases executed by v in considered time interval $[T(v), T(v) + (x+2) \cdot t_{ph}]$. When we multiply the failure probabilities (the successes in epochs are independent) we obtain the failure probability of at most 2^{-x}.

□

In Lemma 6 we showed local bounds on the difference between the reception times of adjacent vertices. Using the Lemma we can show a global bound on the total number of step until all the nodes receive the message.

Theorem 7. *Algorithm* GD-Broadcast *completes broadcast*

1. *in time $O((D + \log n)\log n)$,*

2. *using* $O(\log n/\log\log n)$ *energy per station,*

3. *with probability at least* $1-2/n$.

Proof. Fix any $v \in V$. We want to bound the time until v receives the message. Take the shortest path $\mathcal{P} = (u, v_1, \ldots, v_{D'}, v)$ connecting the originator of the message u, to v. The length of the path satisfies $D' + 1 \leq D$. We have:

$$T(v) = T(v) - T(v_{D'}) + \sum_{i=2}^{D'} T(v_i) - T(v_{i-1}) + T(v_1). \tag{2}$$

Thus let us denote $X_i = |T(v_i) - T(v_{i-1})|$, for $i = 2, \ldots, D'$ and $X_{D'+1} = |T(v) - T(v_{D'})|$ and $X_1 = |T(v_1)|$. Then by (2) $T_v \leq \sum_{i=1}^{D'+1} X_i$. By the definition of the algorithm, the variables X_i are independent. Moreover, we can use Lemma 6 to bound their values. By Lemma 6 we get that with probability at least $1 - n^{-2}$ each variable X_i is at most $(2\lceil \log n \rceil + 2) \cdot t_{ph}$. By the union bound we have that with probability at least $1 - n^{-1}$ all variables X_i satisfy this condition. We take a sequence of $D' + 1$ independent variables $Y_i \sim \mathsf{Geo}(1/2)$. Observe that by Lemma 6 each variable $(Y_i + 2) \cdot t_{ph}$ stochastically dominates X_i (conditioned that $X_i \leq (2\lceil \log n \rceil + 2) \cdot t_{ph}$). If $Y = \sum_{i=1}^{D+2\log n} Y_i$ then by Chernoff bound for geometric variables (see e.g., [2, Theorem 1.14]) we have:

$$\mathbf{Pr}\left[Y \geq 4 \cdot \mathbf{E}\left[X \right] \right] \leq n^{-2}.$$

Hence with probability at least $1 - n^{-2} - n^{-1}$ we have $\sum_{i=1}^{D'} X_i \leq t_{ph} \cdot 6(D + 2\log n)$ which completes the proof of 1 and 3. The energy complexity follows directly from the fact that the energy used by each station in ll consecutive phases is always constant and each station participates in $O(\log n/\log\log n)$ phases.

□

4 Lower bound

Our algorithm BB-Broadcast had a surprisingly large multiplicative factor $n^{1/\varphi}$. In this section we want to show that such a factor is sometimes necessary. We want to show that time $\Omega(n^{1/\varphi} \cdot \varphi)$ is needed for any algorithm using energy φ.

Theorem 8. *For any randomized broadcast algorithm $\mathcal{A}$ successful with probability at least $(1 - e^{-1})/2$ in all multi-hop radio networks there exists a graph G with constant diameter and n nodes, such that if T is the runtime and E is the energy used by the algorithm then the expected value of $E \cdot \log(T/E)$ is $\Omega(\log n)$.*

Proof. We want to prove the theorem using Yao's minimax principle [29]. Take any deterministic algorithm $\mathcal{A}$ and consider any fixed n. We define a family of graphs $\mathcal{G}$ on n vertices. For simplicity assume that $n-1$ is divisible by 2. For any $G_\pi \in \mathcal{G}$ we have $G_\pi = (\{u\} \cup S \cup X, E_s \cup E_\pi)$, where $|S| = |X| = \frac{1}{2}(n-1)$. Node u is the originator of the message in each $G_\pi \in \mathcal{G}$. Also in each $G_\pi \in \mathcal{G}$, set E_s is defined as $E_s = \{(u, s) : s \in S\}$ (i.e., vertices $\{u\} \cup S$ are forming a star with u at its center). Graphs from $\mathcal{G}$ differ on the remaining edges from E_π in the following way. Let $S = \{s_1, s_2, \ldots, s_{(n-1)/2}\}$ and $X = \{x_1, x_2, \ldots, x_{(n-1)/2}\}$. Take the set Π of all permutations $\pi : [(n-1)/2] \to [(n-1)/2]$, such that $\pi(i) \neq i$, for any $i \in [(n-1)/2]$. For all $\pi \in \Pi$ we define E_π as $\{(s_i, x_i), (x_i, s_{\pi(i)}) : i \in \{1, 2, \ldots, (n-1)/2\}\}$. Now consider algorithm $\mathcal{A}$ on graph G_π taken uniformly at random from $\mathcal{G}$. Let T denote the time of the algorithm and E denote maximum energy used by the stations. Consider the total number of possible broadcasting patterns of length T with at most E transmissions:

$$\alpha(T, E) = \sum_{i=0}^{E} \binom{T}{i} \leq \sum_{i=0}^{k} \frac{N^i}{i!} = \sum_{i=0}^{E} \frac{E^i}{i!} \cdot \left(\frac{N}{E}\right)^i \leq \left(\frac{N}{E}\right)^E \sum_{i=0}^{\infty} \frac{E^i}{i!} = \left(\frac{eT}{E}\right)^E.$$

Now, assume that in algorithm $\mathcal{A}$, the last expression is upper bounded by $n/100$. Then also $\alpha(t, x) \leq n/100$. Assume that $n/100$ is an integer. Since there are at most $n/100$ broadcasting patterns then at least $\frac{1}{2}(n-1) - \frac{n}{10} \geq \frac{1}{3}n$ stations from S which have the same pattern as at least 10 other stations from S. Call the set of these $\frac{1}{3}n$ stations $\hat{S}$. We want to lower bound the probability that two stations from $\hat{S}$ have a common neighbor in X. Set $\mathcal{G}$ is defined in such a way that in a graph $G \in \mathcal{G}$ chosen uniformly at random, for each $s \in S$, its corresponding $\pi(s)$ can be seen as a vertex taken uniformly from $S \setminus \{s\}$. Thus, for any $s \in \hat{S}$, the probability that $p(s)$ is using different broadcasting pattern is at most $1 - 5/(n-1)$. Hence with probability at most,

$$\left(1 - \frac{5}{n-1}\right)^{n/3} \leq 1/e,$$

for each station $s \in S$ its corresponding $p(s)$ station uses a different pattern. Hence under the chosen probability distribution over the set of graphs, with probability at least $1 - e^{-1}$, some node does not receive the message. Hence if we define as the cost of the algorithm the expression $(eT/E)^E$ then its expected value is $\Omega(n)$. By Yao's principle [29, Theorem 3] for Monte Carlo algorithms, for any randomized algorithm with error probability at most $(1 - e^{-1})/2$ there exists graph $G \in \mathcal{G}$ such that the expected value of $(eT/E)^E$ is $\Omega(n)$.

□

The following Corollary lower bounds time complexity of any algorithm using the asymptotically same energy as the algorithms presented in Section 3.

Corollary 9. *Any randomized algorithm completing broadcast in any graph with probability at least* $(1 - e^{-1})/2$

1. *using energy at most* φ *needs expected time* $\Omega(n^{1/\varphi} \cdot \varphi)$,
2. *using energy at most* $\frac{\log n}{c \log \log n}$ *for any constant* c *needs expected time* $\Omega\left(\frac{\log^{c+1} n}{\log \log n}\right)$.

This shows that for graphs with constant diameter our algorithms achieve almost optimal tradeoff between time and energy. This also shows that our GD-Broadcast is asymptotically optimal in terms of energetic efficiency among all algorithms that have time polylogarithmic in n.

5 Open problems

A very interesting open problem is whether it is possible to generalize the lower bound for any value of D. Our current results do not rule out an algorithm with energy $O(\varphi)$ and time $\widetilde{O}(D + n^{1/\varphi})$. A second problem would be to develop a time and energy optimal algorithm working in time $O(D \log \frac{n}{D} + \log^2 n)$ and using energy $O(\log n / \log \log n)$.

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Appendix

Below we provide analysis of energy-complexity of classical, fast, randomized broadcasting protocols. We do not provide full descriptions of protocols, but only information necessary for demonstrating lower bound on number of transmissions that is necessary for establishing energy complexity.

Energy-complexity of Bar-Yehuda – Goldreich – Itai protocol

Idea of the protocol This protocol was presented in [3] with expected execution time $O(D\log n + \log^2 n)$.

```
repeat
    at most k times (but at least once!);
    send message M to all neighbors;
    x ← 0 or 1 with equal probability;
until x = 0;
```

Algorithm 5: Original Decay(k) from [3]

In the course of the protocol execution for unknown topology (and at least constant probability of successful delivering messages m to all nodes) the procedure Decay($2\lceil\log n\rceil$) is repeated $\Theta(\log(n))$ times by each of n stations.

Proof of Fact 1

Proof. To prove this Fact we need to prove that the expected maximal number of transmissions over all stations is $\Theta(\log n \log\log n)$.

Definition 1. *A random variable X has **k-truncated geometric distribution** $\mathcal{G}_{1/2,k}$ if $\mathbf{Pr}[X = i] = 1/2^i$ for $1 \le i < k$, $\mathbf{Pr}[X = k] = 1/2^{k-1}$ and 0 otherwise.*

In Broadcast algorithm from [3], each station executes $t = \lceil\log n/\varepsilon\rceil$ times the Decay procedure, where ε is shown to be the failure probability of the algorithm. Each execution requires a random number of transmissions governed by k-truncated geometric distribution $\mathcal{G}_{1/2,k}$ for $k = 2\lceil\log n\rceil$. That is, let $X^{(i)}_{j,k}$ be the number of transmissions during j-th Decay executed by i-th station. Thus the total energy expenditure of the i-th station is $S^{(i)}_k = X^{(i)}_{1,k} + \ldots + X^{(i)}_{t,k}$. Note that $t = 2\lceil\log n/\varepsilon\rceil = \Theta(\log(n))$. Thus the energetic complexity of the algorithm is $\mathbf{E}[S_k] = \mathbf{E}\left[\max\{S^{(1)}_k, \ldots, S^{(n)}_k\}\right]$.

To the best of our knowledge there are no precise results for finding the maximum of such random variables. Note however that in the case of $\mathcal{G}_{1/2,\infty}$, i.e. maximum of sums of regular geometric distribution (a.k.a. negative binomial distribution), precise asymptotic is given by Grabner and Prodinger in [15].

Let S_∞ be the random variable S_k with $k = \infty$. In that case all $X^{(i)}_{j,\infty} \sim \mathcal{G}_{1/2,\infty}$. Immediately from Theorem 1 in [15] we get:

Lemma 10. $\mathbf{E}[S_\infty] = 2\log n \log\log n + \log n + o(\log n)$.

Let us investigate $\mathbf{E}[S_k]$. Let T_k be a binary random variable equal 1 if $X^{(i)}_{\infty,j} \le k$ for all i,j and 0 otherwise. One can observe that

$$\mathbf{E}[S_\infty | T_k = 1] \le \mathbf{E}[S_k] . \tag{3}$$

Let us note that:

$$\mathbf{E}[S_\infty] = \mathbf{E}[S_\infty|T_k = 1]\,\mathbf{Pr}[T_k = 1] + \mathbf{E}[S_\infty|T_k = 0]\,\mathbf{Pr}[T_k = 0] \ ,$$

$$\mathbf{E}[S_\infty|T_k = 1]\,\mathbf{Pr}[T_k = 1] = \mathbf{E}[S_\infty] - \mathbf{E}[S_\infty|T_k = 0]\,\mathbf{Pr}[T_k = 0] \ .$$

Since $t = 2\lceil \log n/\varepsilon \rceil, k = 2\lceil \log \Delta \rceil$ one can observe, assuming that $1/\varepsilon = O(poly(n))$:

$$\mathbf{E}[S_\infty|T_k = 0] \le \mathbf{E}[S_\infty] + t \cdot k = 4\log^2 n + O(\log n) \ .$$

Moreover:

$$\begin{aligned}\mathbf{Pr}[T_k = 0] = \mathbf{Pr}\left[(\exists i,j)\ s.t.\ X^{(i)}_{j,\infty} > k\right] \\ < n \cdot t \cdot \mathbf{Pr}\left[X^{(1)}_{1,\infty} > k\right] \\ < \frac{2n \cdot \log n/\varepsilon}{2^k} \ .\end{aligned}$$

Since we assumed that $k \ge 2\log n$ thus $\mathbf{Pr}[T_k = 0] = o\left(\frac{1}{\log n}\right)$ and we get:

$$\begin{aligned}\mathbf{E}[S_\infty|T_k = 1]\,\mathbf{Pr}[T_k = 1] \ge 2\log n \log\log n + \log n + o(\log n) + \\ - (\log^2 n + 2\log n \log\log n + O(\log n)) \cdot o\left(\frac{1}{\log n}\right) .\end{aligned}$$

Thus $\mathbf{E}[S_\infty|T_k = 1] = \Theta(\log n \log\log n)$ and using equation (3) we get that $\mathbf{E}[S_k] = \Theta(\log n \log\log n)$, which completes the proof. □

Energy-complexity of Czumaj-Rytter protocol

Idea of the protocol Let D be the diameter of the graph uses a common sequence $\mathcal{J} = (J_1, J_2, \ldots)$. It has a following property. It contains $\Theta(D)$ subsequences $(1, \ldots, \log(n/D))$ in the first $T = D\log(n/D)$ positions. Note that this sequence is fixed for all the stations.

In the course of the algorithm each station transmits in $T = \Theta(D\log(n/D))$ consecutive rounds with probability. In the i-th round the station transmits with probability 2^{-J_i} .

Proof of Fact 2

Proof. Note that in T first elements of the sequence $\mathcal{J}$ there are $\Theta(D)$ ones. This means that each station transmits with probability $1/2$ in $\Theta(D)$ rounds. Thus the expected number of transmission is $\Omega(D)$. Clearly, this implies that the expected maximal number over all stations is $\Omega(D)$ as well. o complete the proof it is enough to note that by the assumption about D the expected maximal number of transmissions is also $\Omega(\log^3 n)$. □

Energy-complexity of Kowalski-Pelc protocol

Idea of the protocol The idea of the protocol is similar to Czumaj–Rytter protocol for graphs with large diameter D. Otherwise, if $D = o(n^{2/3})$ Bar-Yehuda–Goldreich–Itai randomized protocol from [3] is launched. For large D a subprotocol *Stage* is executed, wherein each station transmits with probabilities $1/2^l$ for $l = 0, \ldots, \log(N/D)$.

Proof of Fact 3

Proof. The case for small D follows directly from Fact 1. To prove the second case it is enough to note that each station has to execute D times the *Stage* procedure. Each time the *Stage* launched, the station transmits at least once. Since $D = \Omega(n^{2/3})$ the proof is completed. □